

Catalan Moments

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Abstract

This paper is essentially devoted to the study of some interesting relations among the well known operators $I^{(x)}$ (the interpolated Invert), $L^{(x)}$ (the interpolated Binomial) and Revert (that we call η).

We prove that $I^{(x)}$ and $L^{(x)}$ are conjugated in the group $\Upsilon(R)$. Here R is a commutative unitary ring. In the same group we see that η transforms $I^{(x)}$ in $L^{(-x)}$ by conjugation. These facts are proved as corollaries of much more general results.

Then we carefully analyze the action of these operators on the set \mathcal{R} of second order linear recurrent sequences. While $I^{(x)}$ and $L^{(x)}$ transform \mathcal{R} in itself, η sends \mathcal{R} in the set of moment sequences $\mu_n(h, k)$ of particular families of orthogonal polynomials, whose weight functions are explicitly computed.

The moments come out to be generalized Motzkin numbers (if $R = \mathbb{Z}$, the Motzkin numbers are $\mu_n(-1, 1)$). We give several interesting expressions of $\mu_n(h, k)$ in closed forms, and one recurrence relation.

There is a fundamental sequence of moments, that generates all the other ones, $\mu_n(0, k)$. These moments are strongly related with Catalan numbers. This fact allows us to find, in the final part, a new identity on Catalan numbers by using orthogonality relations.

1 A group acting on sequences

Definition 1.1.

$$\mathcal{S}(R) = \{A = \{a_n\}_{n=0}^{+\infty} : \forall n \ a_n \in R, a_0 = 1\}$$

where R is a commutative unitary ring.

Now we embed $\mathcal{S}(R)$ in $R[[t]]$ in this way

$$(1) \quad \forall A \in \mathcal{S}(R) \quad \lambda(A) = \sum_{n=0}^{+\infty} a_n t^{n+1}.$$

In $R[[t]]$ is naturally defined the series composition \circ

$$\sum_{n=0}^{+\infty} a_n t^{n+1} \circ \sum_{k=0}^{+\infty} b_k t^{k+1} = \sum_{n=0}^{+\infty} a_n \left(\sum_{k=0}^{+\infty} b_k t^{k+1} \right)^{n+1}.$$

Then we may induce the operation \bullet in $\mathcal{S}(R)$:

Definition 1.2.

$$\forall A, B \in \mathcal{S}(R) \quad A \bullet B = \lambda^{-1}(\lambda(A) \circ \lambda(B)) \quad .$$

Of course $(\mathcal{S}(R), \bullet)$ is a group.

Observation 1.3. If $R = \mathbb{F}_q$, this is the *Nottingham group* over \mathbb{F}_q [4] .

From every element $A \in \mathcal{S}(R)$ two operators rise: the left multiplication \mathcal{L}_A and the right multiplication \mathcal{R}_A .

Definition 1.4.

$$\forall B \in \mathcal{S}(R) \quad \mathcal{L}_A(B) = A \bullet B$$

$$\forall B \in \mathcal{S}(R) \quad \mathcal{R}_A(B) = B \bullet A \quad .$$

We also consider the following two special operators: η , often called **Revert**, and ε the alternating sign operator :

Definition 1.5.

$$\forall B \in \mathcal{S}(R) \quad \eta(B) = B^{-1}$$

$$\forall B = \{b_n\}_{n=0}^{+\infty} \in \mathcal{S}(R) \quad \varepsilon(B) = \{(-1)^n b_n\}_{n=0}^{+\infty} .$$

Plainly

Property 1.6.

$$(2) \quad \forall A, B \in \mathcal{S}(R) \quad \eta(A \bullet B) = \eta(B) \bullet \eta(A)$$

$$(3) \quad \forall A, B \in \mathcal{S}(R) \quad \varepsilon(A \bullet B) = \varepsilon(A) \bullet \varepsilon(B) \quad .$$

In other words, the inversion η is an anti-isomorphism of $\mathcal{S}(R)$, and the alternating sign ε is an isomorphism of $\mathcal{S}(R)$.

Observation 1.7. The operator η is especially important. If $a = \{a_n\}_{n=0}^{+\infty}$, $b = \{b_n\}_{n=0}^{+\infty}$ and $\eta(a) = b$, then we have the relations

$$(4) \quad \begin{cases} u = u(t) = \sum_{n=0}^{+\infty} a_n t^{n+1} \\ t = t(u) = \sum_{n=0}^{+\infty} b_n u^{n+1} \end{cases} \quad \text{inverse series of } u \quad .$$

The operators \mathcal{L}_A , \mathcal{R}_A , η , ε , are invertible and can be compounded by applying one after the other (by the usual operation \circ). They generate a group, that we call $\Upsilon(R)$. The group $\Upsilon(R)$ acts on $\mathcal{S}(R)$.

Property 1.8. *For every ring R the following are true*

$$(5) \quad \eta = \eta^{-1}$$

$$(6) \quad \varepsilon = \varepsilon^{-1}$$

$$(7) \quad \eta \circ \varepsilon = \varepsilon \circ \eta$$

$$(8) \quad \forall A, B \in \mathcal{S}(R) \quad \mathcal{L}_A \circ \mathcal{R}_B = \mathcal{R}_B \circ \mathcal{L}_A$$

$$(9) \quad \forall A \in \mathcal{S}(R) \quad \mathcal{L}_A \circ \eta = \eta \circ \mathcal{R}_{A^{-1}}$$

$$(10) \quad \forall A \in \mathcal{S}(R) \quad \eta \circ \mathcal{L}_A = \mathcal{R}_{A^{-1}} \circ \eta \quad .$$

Proof.

(5) and (6) follow from definition.

(7) Let $d = \eta(\varepsilon(a))$. Because $\varepsilon(a) = \{(-1)^n a_n\}_{n=0}^{+\infty}$, (4) becomes

$$(11) \quad \begin{cases} u = u(t) = \sum_{n=0}^{+\infty} (-1)^n a_n t^{n+1} \\ t = t(u) = \sum_{n=0}^{+\infty} d_n u^{n+1}. \end{cases}$$

But (11(i)) can be rewritten as

$$-u = \sum_{n=0}^{+\infty} a_n (-t)^{n+1}$$

and if $b = \eta(a)$ then

$$t = \sum_{n=0}^{+\infty} (-1)^n b_n u^{n+1}$$

comparing this result with (11(ii)) we obtain $d = \varepsilon(b)$.

(8)

$$\forall A, B, C \in \mathcal{S}(R)$$

$$(\mathcal{L}_A \circ \mathcal{R}_B)(C) = \mathcal{L}_A(C \bullet B) = A \bullet C \bullet B = \mathcal{R}_B(A \bullet C) = \mathcal{R}_B(\mathcal{L}_A(C)) = (\mathcal{R}_B \circ \mathcal{L}_A)(C) \quad .$$

(9)

$$\forall A, B \in \mathcal{S}(R) \quad (\mathcal{L}_A \circ \eta)(B) = A \bullet B^{-1}$$

and

$$\forall A, B \in \mathcal{S}(R) \quad (\eta \circ \mathcal{R}_{A^{-1}})(B) = (B \bullet A^{-1})^{-1} = A \bullet B^{-1}.$$

(10) To prove this we do operator composition with η both in the front and the back of each side of (9) . \square

Let us pose $\gamma = \eta \circ \varepsilon$ and $X(x) = \{x^n\}_{n=0}^{+\infty}$, with $x \in R$.

Of course $X(x) \in \mathcal{S}(R)$ and both $\mathcal{L}_{X(x)}$ and $\mathcal{R}_{X(x)}$ are in $\Upsilon(R)$.

Plainly

$$(12) \quad \begin{cases} \text{the generating function of } X(x) \text{ is } \frac{1}{1-xt} \\ \lambda(X(x)) = \sum_{n=0}^{+\infty} x^n t^{n+1} = \frac{t}{1-xt}. \end{cases}$$

We have

Property 1.9.

(13)

$$\mathcal{L}_{X(x)} \circ \varepsilon = \varepsilon \circ \mathcal{L}_{X(-x)}$$

(14)

$$\mathcal{R}_{X(x)} \circ \varepsilon = \varepsilon \circ \mathcal{R}_{X(-x)}$$

(15)

$$\gamma = \gamma^{-1}$$

(16)

$$\gamma \circ \mathcal{L}_{X(x)} \circ \gamma^{-1} = \mathcal{R}_{X(x)}$$

(17)

$$\gamma \circ \mathcal{R}_{X(x)} \circ \gamma^{-1} = \mathcal{L}_{X(x)}.$$

Proof.

$$(13) \quad \forall A \in \mathcal{S}(R)$$

$$(\mathcal{L}_{X(x)} \circ \varepsilon)(A) = X(x) \bullet \varepsilon(A) = \varepsilon(X(-x)) \bullet \varepsilon(A) = \varepsilon(X(-x) \bullet A) = (\varepsilon \circ \mathcal{L}_{X(-x)})(A) \quad .$$

(14) Same proof as for (13).

(15) γ is the composition of two commuting involutions.

$$(16) \quad \forall A \in \mathcal{S}(R)$$

$$(\gamma \circ \mathcal{L}_{X(x)})(A) = (\eta \circ \varepsilon)(X(x) \bullet A) = \eta(X(-x) \bullet \varepsilon(A)) = \gamma(A) \bullet X(x) = (\mathcal{R}_{X(x)} \circ \gamma)(A) \quad .$$

(17) Same proof as for (16).

□

Let us recall the well known operators Invert and Binomial.

Definition 1.10. The operator I maps the sequence $A = \{a_n\}_{n=0}^{+\infty}$ in $B = \{b_n\}_{n=0}^{+\infty}$ where B has generating function:

$$\sum_{n=0}^{\infty} b_n t^n = \frac{\sum_{n=0}^{+\infty} a_n t^n}{1 - t \sum_{n=0}^{\infty} a_n t^n} \quad .$$

Definition 1.11. The operator L maps the sequence $A = \{a_n\}_{n=0}^{+\infty}$ in $B = \{b_n\}_{n=0}^{+\infty}$ where

$$b_n = \sum_{k=0}^n \binom{n}{k} a_k \quad .$$

These operators can be *iterated* [11] and *interpolated* [1] becoming $I^{(x)}$, $L^{(y)}$ in this way:

Definition 1.12. Given $x \in R$ $I^{(x)}$ is called **Invert** interpolated operator. By definition $I^{(x)}(A) = P = \{p_n(x)\}_{n=0}^{+\infty}$ where P is the sequence having generating function

$$(18) \quad \mathbf{P}(t) = \sum_{n=0}^{+\infty} p_n(x) t^n = \frac{\sum_{n=0}^{+\infty} a_n t^n}{1 - xt \sum_{n=0}^{+\infty} a_n t^n} \quad .$$

Definition 1.13. Given $y \in R$ $L^{(y)}$ is called **Binomial** interpolated operator. By definition

$$(19) \quad L^{(y)}(A) = \left\{ l_n = \sum_{j=0}^n \binom{n}{j} y^{n-j} a_j \right\}_{n=0}^{+\infty}.$$

The exponential generating function of $l = \{l_n\}_{n=0}^{+\infty}$ is:

$$(20) \quad \mathcal{L}(t) = \sum_{n=0}^{+\infty} l_n \frac{t^n}{n!} = \sum_{n=0}^{+\infty} \sum_{j=0}^n \frac{(yt)^{n-j}}{(n-j)!} \frac{a_j t^j}{j!} = \exp(ty) \mathcal{A}(t)$$

being

$$\exp(ty) = \sum_{n=0}^{+\infty} \frac{(ty)^n}{n!} \quad \mathcal{A}(t) = \sum_{n=0}^{+\infty} \frac{a_n t^n}{n!}$$

so that (recalling that $a_0 = 1$) we have the ordinary generating function

$$(21) \quad \mathbf{L}(t) = \frac{1}{t} A\left(\frac{t}{1-ty}\right)$$

with $A(t) = \sum_{n=0}^{+\infty} a_n t^{n+1}$.

The following facts are immediate consequences of (12).

Property 1.14.

$$(22) \quad \forall x \in R \quad \eta(X(x)) = X^{-1}(x) = \{(-x)^n\}_{n=0}^{+\infty} = X(-x) = \varepsilon(X(x))$$

$$(23) \quad \forall x, y \in R \quad X(x) \bullet X(y) = X(x+y) \quad .$$

From their definitions it is not apparent that the operators $I^{(x)}$ and $L^{(x)}$ are strongly related. Indeed we are going to prove that they are, respectively, the left and the right multiplication by $X(x)$ in the group $\mathcal{S}(R)$.

Theorem 1.15.

$$(24) \quad I^{(x)} = \mathcal{L}_{X(x)}$$

$$(25) \quad L^{(x)} = \mathcal{R}_{X(x)} \quad .$$

Proof.

(24) Let $B = X(x) \bullet A$, then

$$\lambda(B) = \lambda(X(x)) \circ \lambda(A) = \sum_{n=0}^{+\infty} x^n \left(\sum_{k=0}^{+\infty} a_k t^{k+1} \right)^{n+1} = \sum_{k=0}^{+\infty} a_k t^{k+1} \left(\sum_{n=0}^{+\infty} \left(x \sum_{k=0}^{+\infty} a_k t^{k+1} \right)^n \right) = \frac{\sum_{k=0}^{+\infty} a_k t^{k+1}}{1 - x \sum_{k=0}^{+\infty} a_k t^{k+1}}$$

so $B = I^{(x)}(A)$ from (18) and (1).

(25) Let $C = A \bullet X(x)$, then

$$\lambda(C) = \lambda(A) \circ \lambda(X(x)) = \sum_{n=0}^{+\infty} a_n \left(\sum_{k=0}^{+\infty} x^k t^{k+1} \right)^{n+1} = \sum_{n=0}^{+\infty} a_n \left(\frac{t}{1 - xt} \right)^{n+1}$$

so $C = L^{(x)}(A)$ from (21) and (1).

□

From Theorem 1.15 and the previous properties we obtain

Theorem 1.16. *Let Id be the identity operator and $x, y \in R$. For the interpolated Invert and Binomial operators the following are true:*

$$I^{(x)} \circ I^{(-x)} = Id \quad L^{(x)} \circ L^{(-x)} = Id$$

$$I^{(x)} \circ I^{(y)} = I^{(x+y)} \quad L^{(x)} \circ L^{(y)} = L^{(x+y)}$$

$$I^{(x)} \circ \varepsilon = \varepsilon \circ I^{(-x)} \quad L^{(x)} \circ \varepsilon = \varepsilon \circ L^{(-x)}$$

$$I^{(x)} \circ L^{(y)} = L^{(y)} \circ I^{(x)}$$

$$I^{(x)} \circ \eta = \eta \circ L^{(-x)} \quad \eta \circ I^{(x)} = L^{(-x)} \circ \eta$$

$$\gamma \circ I^{(x)} \circ \gamma^{-1} = L^{(x)} \quad \gamma \circ L^{(x)} \circ \gamma^{-1} = I^{(x)} \quad .$$

So we have seen, by the way, that the operators $I^{(x)}$ and $L^{(x)}$ are *conjugated* in the group $\Upsilon(R)$!

2 Action on linear recurrent sequences of order 2

In this section we analyze the action of $I^{(x)}$ and $L^{(x)}$ on the particular subset of $\mathcal{S}(R)$ formed by linear recurrent sequences of order 2 (starting with 1).

Definition 2.1.

$$\mathcal{R}(R) = \{\mathcal{W}(1, b, h, k) : b, h, k \in R\}$$

where

$$\mathcal{W}(1, b, h, k) = \{\mathcal{W}_n(1, b, h, k)\}_{n=0}^{+\infty}$$

satisfies the recurrence $\forall n \geq 2$

$$(26) \quad \begin{cases} \mathcal{W}_0(1, b, h, k) = 1 \\ \mathcal{W}_1(1, b, h, k) = b \\ \mathcal{W}_n(1, b, h, k) = h\mathcal{W}_{n-1}(1, b, h, k) - k\mathcal{W}_{n-2}(1, b, h, k) \quad \forall n \geq 2 \end{cases} .$$

$I^{(x)}$ and $L^{(x)}$ map $\mathcal{R}(R)$ into itself in the following way

Theorem 2.2. $\forall x, y \in R$ we have

$$(27) \quad I^{(x)}(\mathcal{W}(1, b, h, k)) = \mathcal{W}(1, b + x, h + x, (h - b)x + k)$$

$$(28) \quad L^{(y)}(\mathcal{W}(1, b, h, k)) = \mathcal{W}(1, b + y, h + 2y, y^2 + hy + k)$$

$$(29) \quad C^{(x,y)}(\mathcal{W}(1, b, h, k)) = \mathcal{W}(1, b + y + x, h + x + 2y, y^2 + hy + k + (h - b)x + xy)$$

where $C^{(x,y)} = I^{(x)} \circ L^{(y)} = L^{(y)} \circ I^{(x)}$.

Proof. The generating function of $\mathcal{W}(1, b, h, k)$ is

$$(30) \quad \mathbf{W}(t) = \frac{1 + (b - h)t}{1 - ht + kt^2}.$$

If we substitute $\mathbf{W}(t)$ in (18), and compute $\mathbf{P}(\mathbf{W}(t))$, we find

$$\frac{1 + (b - h)t}{1 - (h + x)t + (k + (h - b)x)t^2}.$$

This proves (27). In the ring $R[z]/(z^2 - hz + k)$ we pose $\alpha_1 = z$ and $\alpha_2 = h - z$ (the roots of $z^2 - hz + k$). Then we have $\mathcal{W}_n(1, b, h, k) = p\alpha_1^n + q\alpha_2^n$.

Substituting the sequence $\mathcal{W}(1, b, h, k)$ to the sequence A in (19) we obtain

$$\begin{aligned} l_n &= \sum_{i=0}^n \binom{n}{i} y^{n-i} a_i = \sum_{i=0}^n \binom{n}{i} y^{n-i} (p\alpha_1^i + q\alpha_2^i) = \\ &= p \sum_{i=0}^n \binom{n}{i} y^{n-i} \alpha_1^i + q \sum_{i=0}^n \binom{n}{i} y^{n-i} \alpha_2^i = p(y + \alpha_1)^n + q(y + \alpha_2)^n . \end{aligned}$$

Then posing $y + \alpha_1 = R$ and $y + \alpha_2 = S$, observing that

$$R + S = 2y + \alpha_1 + \alpha_2 = h + 2y \quad RS = y^2 + (\alpha_1 + \alpha_2)y + \alpha_1\alpha_2 = y^2 + hy + k$$

where we used $\alpha_1 + \alpha_2 = h$ and $\alpha_1\alpha_2 = k$, we have:

$$\begin{aligned} l_n &= pR^n + qS^n + pR^{n-1}S + qS^{n-1}R - pR^{n-1}S - qS^{n-1}R = \\ &= R(pR^{n-1} + qS^{n-1}) + S(pR^{n-1} + qS^{n-1}) - RS(pR^{n-2} + qS^{n-2}) = \\ &= (R+S)(pR^{n-1} + qS^{n-1}) - RS(pR^{n-2} + qS^{n-2}) = (h+2y)l_{n-1} - (y^2 + hy + k)l_{n-2} . \end{aligned}$$

This proves (28).

(29) follows at once from (27) and (28). \square

Observation 2.3. An important subset $\mathcal{F} \subset \mathcal{R}(R)$ consists of sequences

$$F(h, k) = \mathcal{W}(1, h, h, k) = \{1, h, h^2 - k, \dots\}$$

(31)

$$\begin{cases} \mathcal{W}_0(1, h, h, k) = 1 \\ \mathcal{W}_1(1, h, h, k) = h \\ \mathcal{W}_n(1, h, h, k) = h\mathcal{W}_{n-1}(1, h, h, k) - k\mathcal{W}_{n-2}(1, h, h, k) \quad \forall n \geq 2 \end{cases} .$$

They are a subset of generalized Fibonacci sequences.

From Theorem 2.2 we can define a polynomial sequence $\mathcal{P}(h, k, x)$ as follows

$$(32) \quad \mathcal{P}(h, k, x) = \{P_n(h, k, x)\}_{n=0}^{+\infty} = I^{(x)}(F(h, k)) = F(h + x, k)$$

and we can observe that

$$(33) \quad I^{(h)}(\mathcal{W}(1, 0, 0, k)) = I^{(h)}(F(0, k)) = F(h, k).$$

These relations, as we will see in the next section, show a connection between \mathcal{F} and orthogonal polynomials. They also help us to prove what Bacher [1] observes about arithmetical properties of $P_n(h, k, x)$.

Proposition 2.4. $\forall m, n$ such that $m|n$ then $P_{m-1}(h, k, x) | P_{n-1}(h, k, x)$.

Proof.

(32) gives the recurrence relation

$$(34) \quad \begin{cases} P_0(h, k, x) = 1 \\ P_1(h, k, x) = h + x \\ P_n(h, k, x) = (h + x)P_{n-1}(h, k, x) - kP_{n-2}(h, k, x) \quad \forall n \geq 2 \end{cases}$$

from which

$$P_n(h, k, x) = \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2}$$

where α_1 and α_2 are the roots of the characteristic polynomial

$$t^2 - (h + x)t + k = 0 \quad .$$

Thus

$$\frac{P_{n-1}(h, k, x)}{P_{m-1}(h, k, x)} = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1^m - \alpha_2^m}$$

and if $m|n$ then $P_{m-1}(h, k, x) | P_{n-1}(h, k, x)$.

□

Finally we can find a couple of relations on sequences $\mathcal{W}(1, b, h, k)$ involving the η operator.

Corollary 2.5. *For all sequences $\mathcal{W}(1, b, h, k)$ we have*

$$(35) \quad I^{(x)}(\eta(\mathcal{W}(1, b, h, k))) = \eta(W(1, b - x, h - 2x, x^2 - hx + k))$$

$$(36) \quad L^{(x)}(\eta(\mathcal{W}(1, b, h, k))) = \eta(W(1, b - x, h - x, (b - h)x + k)) \quad .$$

Proof. The proof is obvious from Theorem (1.16).

□

3 Moments generating function

From now on we shall pose $R = \mathbb{C}$.

We know from (32) that $I^{(x)}$, applied to elements in \mathcal{F} , gives rise to polynomial sequence $\mathcal{P}(h, k, x) = \{P_n(h, k, x)\}_{n=-1}^{+\infty}$ (where indexes have been changed in (32) for convenience in calculation), with recurrence relation

$$(37) \quad \begin{cases} P_{-1}(h, k, x) = 0 \\ P_0(h, k, x) = 1 \\ P_n(h, k, x) = (x + h)P_{n-1}(h, k, x) - kP_{n-2}(h, k, x) \quad \forall n \geq 1 \end{cases} .$$

From Favard's theorem ([5], page 21) this recurrence relation, when $k \neq 0$, is also the one for orthogonal polynomials having a proper moment functional. If $h = 0$ we have $P_n(0, k, x) = E_n(x, k)$ the n -th Dickson polynomial of the second kind [8].

Moreover for the moments sequence $\mu(h, k)$ related to the sequence $\mathcal{P}(h, k, x)$ the following holds :

Theorem 3.1. *The sequence $\mu(h, k)$ has generating function*

$$(38) \quad \mu(t) = \sum_{n=0}^{+\infty} \mu_n t^n = \frac{1 - ht - \sqrt{(1 - ht)^2 - 4kt^2}}{2kt^2} .$$

Proof. From known results about orthogonal polynomials theory [5], the moments generating function $\mu(t)$ is equal to a continued fraction :

$$(39) \quad \mu(t) = \sum_{n=0}^{+\infty} \mu_n t^n = \frac{\lambda_0}{1 + \xi_0 t - \frac{\lambda_1 t^2}{1 + \xi_1 t - \frac{\lambda_2 t^2}{1 + \xi_2 t - \frac{\lambda_3 t^2}{1 + \xi_3 t - \dots}}}} .$$

For $\mathcal{P}(h, k, x)$ we have $\forall n \ \xi_n = -h, \forall n > 1 \ \lambda_n = k, \lambda_0 = \mu_0 = 1$ and (39) becomes

$$(40) \quad \mu(t) = \sum_{n=0}^{+\infty} \mu_n t^n = \frac{1}{1 - ht - \frac{kt^2}{1 - ht - \frac{kt^2}{1 - ht - \frac{kt^2}{1 - ht - \dots}}}} .$$

It can be expressed in closed form posing

$$y = \frac{kt^2}{1 - ht - \frac{kt^2}{1 - ht - \frac{kt^2}{1 - ht - \frac{kt^2}{1 - ht - \dots}}}}$$

and observing that

$$(41) \quad y = \frac{kt^2}{1 - ht - y}$$

and

$$(42) \quad \mu(t) = \frac{1}{1 - ht - y} .$$

Finding y from (41) we obtain

$$y_1 = \frac{1 - ht + \sqrt{(1 - ht)^2 - 4kt^2}}{2}$$

$$y_2 = \frac{1 - ht - \sqrt{(1 - ht)^2 - 4kt^2}}{2}.$$

We have to choose $y = y_2$ because y_1 replaced in (42) gives rise to discontinuity at $t = 0$. With this value for y and a rationalization we easily find the exact form of $\mu(t)$ in (38). \square

The explicit moments values are given by the

Corollary 3.2. *The moments $\mu_n(h, k)$ related to polynomials $\mathcal{P}(h, k, x)$ are equal to*
(43)

$$\mu_n(h, k) = \begin{cases} -\frac{1}{2k} \sum_{j=0}^{\frac{n+1}{2}} \binom{1/2}{n+2-j} \binom{n+2-j}{j} (-2h)^{n+2-2j} (h^2 - 4k)^j & n \text{ odd} \\ -\frac{1}{2k} \sum_{j=0}^{\frac{n+2}{2}} \binom{1/2}{n+2-j} \binom{n+2-j}{j} (-2h)^{n+2-2j} (h^2 - 4k)^j & n \text{ even} \end{cases}$$

where $n \geq 1$ and $\mu_0 = 1$.

Proof. The result follows developing (38):

$$\sqrt{(1 - ht)^2 - 4kt^2} = (1 - 2ht + (h^2 - 4k)t^2)^{1/2} = \sum_{i=0}^{+\infty} \binom{1/2}{i} (-2ht + (h^2 - 4k)t^2)^i$$

and so

$$\begin{aligned} \mu(t) &= \frac{1}{2kt^2} \left(1 - ht - (1 - ht) - \frac{1}{2}(h^2 - 4k)t^2 - \sum_{i=2}^{+\infty} \binom{1/2}{i} (-2ht + (h^2 - 4k)t^2)^i \right) = \\ &= -\frac{1}{4k}(h^2 - 4k) - \frac{1}{2k} \sum_{i=2}^{+\infty} \binom{1/2}{i} \sum_{j=0}^i \binom{i}{j} (-2h)^{i-j} (h^2 - 4k)^j t^{i+j-2}. \end{aligned}$$

Ordering the summation with respect to the degree n of t^n , we observe that the coefficient of t^n for $n = 0$ is $-h^2/2$ and replacing it in $\mu(t)$ expression we have

$$\mu(t) = 1 + \sum_{n=1}^{+\infty} \mu_n(h, k) t^n$$

$$\mu_n(h, k) = \begin{cases} -\frac{1}{2k} \sum_{j=0}^{\frac{n+1}{2}} \binom{1/2}{n+2-j} \binom{n+2-j}{j} (-2h)^{n+2-2j} (h^2 - 4k)^j & \text{for odd } n \\ -\frac{1}{2k} \sum_{j=0}^{\frac{n+2}{2}} \binom{1/2}{n+2-j} \binom{n+2-j}{j} (-2h)^{n+2-2j} (h^2 - 4k)^j & \text{for even } n. \end{cases}$$

\square

Observation 3.3. The moments $\mu_n(h, k)$ are the generalized Motzkin numbers. We will show a combinatorial interpretation of them in Section 6.

4 Weight function

We want to find the weight function $\omega(t)$ of the functional \mathcal{V} related to the sequence $\mathcal{P}(h, k, x)$ (see [5]). So $\mathcal{V}[f]$ will be defined as follows

$$\mathcal{V}[f] = \int_C f(t) d\psi(t)$$

where C will be a suitable integration interval, $\psi(t)$ a distribution such that $\psi'(t) = \omega(t)$. By Stieltjes inversion formula we have

$$(44) \quad \psi(t) - \psi(0) = -\frac{1}{\pi} \lim_{y \rightarrow 0^+} \int_0^t \mathcal{I}m(F(x + iy, h, k)) dx$$

being $z = x + iy \in \mathbb{C}$ and $F(z, h, k) = z^{-1} \mu(z^{-1})$ where $\mu(t)$ is defined by (38); thus

$$(45) \quad F(z, h, k) = \frac{z - h - \sqrt{(z - h)^2 - 4k}}{2k}.$$

We can immediately find the corresponding primitive $\mathcal{F}(z, h, k)$ of $F(z, h, k)$

$$\mathcal{F}(z, h, k) = \frac{1}{2k} \left(\frac{z^2}{2} - hz - \frac{(z - h)}{2} \sqrt{(z - h)^2 - 4k} - 2k \log \left(\sqrt{(z - h)^2 - 4k} - (z - h) \right) \right)$$

where the arbitrary constant has been made equal to 0, without loss of generality.

Now we can study, depending on h, k , the value of

$$(46) \quad \mathcal{I}m \left(\lim_{y \rightarrow 0^+} \mathcal{F}(x + iy, h, k) \right)$$

considering all the parts which summed together give \mathcal{F} :

i.

$$\lim_{y \rightarrow 0^+} \left(\frac{1}{2k} \left(\frac{z^2}{2} - hz \right) \right) = \frac{1}{2k} \left(\frac{x^2}{2} - hx \right)$$

ii.

$$\lim_{y \rightarrow 0^+} \left(\frac{1}{4k} (z - h) \sqrt{(z - h)^2 - 4k} \right) = \frac{1}{4k} (x - h) \sqrt{(x - h)^2 - 4k}$$

iii.

$$\lim_{y \rightarrow 0^+} \left(\log \left(\sqrt{(z - h)^2 - 4k} - (z - h) \right) \right) = \log \left(\sqrt{(x - h)^2 - 4k} - (x - h) \right)$$

remembering the condition $k \neq 0$, we note that :

i) is always real;

ii) is real if $(x - h)^2 - 4k \geq 0$ or $k < 0$, otherwise

$$\lim_{y \rightarrow 0^+} \left(\frac{1}{4k} (z - h) \sqrt{(z - h)^2 - 4k} \right) = \frac{i}{4k} (x - h) \sqrt{4k - (x - h)^2}$$

when $h - 2\sqrt{k} < x < h + 2\sqrt{k}$;

iii) if $k < 0$ or $k > 0$ and $x \notin (h - 2\sqrt{k}, h + 2\sqrt{k})$, $\sqrt{(x - h)^2 - 4k}$ is real, moreover

$$\sqrt{(x - h)^2 - 4k} - (x - h) > 0$$

surely if $k < 0$, while if $k > 0$ and $x \notin (h - 2\sqrt{k}, h + 2\sqrt{k})$ the logarithm is real if $x \in (-\infty, h - 2\sqrt{k})$ and complex if $x \in (h + 2\sqrt{k}, +\infty)$.

In this ultimate case we have

$$\log \left(\sqrt{(x - h)^2 - 4k} - (x - h) \right) = \log \left| \sqrt{(x - h)^2 - 4k} - (x - h) \right| + i\pi \quad .$$

Finally if $k > 0$ and $x \in (h - 2\sqrt{k}, h + 2\sqrt{k})$ then

$$\sqrt{(x - h)^2 - 4k} = i\sqrt{4k - (x - h)^2}$$

and

$$\begin{aligned} \log \left(\sqrt{(x - h)^2 - 4k} - (x - h) \right) &= \log \left(-(x - h) + i\sqrt{4k - (x - h)^2} \right) = \\ &= \log \left| -(x - h) + i\sqrt{4k - (x - h)^2} \right| + i \operatorname{Arg} \left(-(x - h) + i\sqrt{4k - (x - h)^2} \right) \end{aligned}$$

with

$$\operatorname{Arg} \left(-(x - h) + i\sqrt{4k - (x - h)^2} \right) = \begin{cases} -\arctan \left(\frac{\sqrt{4k - (x - h)^2}}{(x - h)} \right) & \text{if } h - 2\sqrt{k} < x < h \\ \frac{\pi}{2} & \text{if } x = h \\ \pi - \arctan \left(\frac{\sqrt{4k - (x - h)^2}}{(x - h)} \right) & \text{if } h < x < h + 2\sqrt{k} . \end{cases}$$

So the limit (46) is zero for $k < 0$ and also for $k > 0$ with $x \in (-\infty, h - 2\sqrt{k})$ while when $k > 0$ and $x \in (h - 2\sqrt{k}, +\infty)$ the limit values are

$$\begin{cases} -\frac{(x - h)\sqrt{4k - (x - h)^2}}{4k} + \arctan \left(\frac{\sqrt{4k - (x - h)^2}}{(x - h)} \right) & \text{if } h - 2\sqrt{k} < x < h \\ -\frac{\pi}{2} & \text{if } x = h \\ -\frac{(x - h)\sqrt{4k - (x - h)^2}}{4k} - \pi + \arctan \left(\frac{\sqrt{4k - (x - h)^2}}{(x - h)} \right) & \text{if } h < x < h + 2\sqrt{k} \\ -\pi & \text{if } x > h + 2\sqrt{k} . \end{cases}$$

This gives, together with (44)

$$(47) \quad \omega(t) = \psi'(t) = \begin{cases} \frac{\sqrt{4k-(t-h)^2}}{2k\pi} & \text{if } h-2\sqrt{k} < t < h+2\sqrt{k} \wedge t \neq h \\ 0 & \text{otherwise .} \end{cases}$$

5 Recurrence relation for $\mu(h, k)$

We know from definition that

$$(48) \quad \mu_n = \mathcal{V}[t^n] = \int_C t^n d\psi(t)$$

and relation (48) becomes, using (47)

$$(49) \quad \mu_n = \int_{h-2\sqrt{k}}^{h+2\sqrt{k}} \frac{t^n \sqrt{4k-(t-h)^2}}{2k\pi} dt \quad .$$

Now we can prove the

Theorem 5.1. *The sequence $\mu(h, k)$ is recurrent with*

$$(50) \quad \begin{cases} \mu_0 = 1 \\ \mu_1 = h \\ \mu_n = \frac{h(2n+1)\mu_{n-1} - (h^2 - 4k)(n-1)\mu_{n-2}}{n+2} \quad \forall n \geq 2 . \end{cases}$$

Proof. $\forall n \geq 2$ we have

$$\begin{aligned} \mu_n &= \int_{h-2\sqrt{k}}^{h+2\sqrt{k}} \frac{t^{n-1}(t-h+h)\sqrt{4k-(t-h)^2}}{2k\pi} dt = \\ &= \int_{h-2\sqrt{k}}^{h+2\sqrt{k}} \frac{t^{n-1}(t-h)\sqrt{4k-(t-h)^2}}{2k\pi} dt + h\mu_{n-1} \end{aligned}$$

using integration by parts we obtain

$$\mu_n = \left[\frac{-t^{n-1}\sqrt{(4k-(t-h)^2)^3}}{6k\pi} \right]_{h-2\sqrt{k}}^{h+2\sqrt{k}} - \int_{h-2\sqrt{k}}^{h+2\sqrt{k}} \frac{-(n-1)t^{n-2}\sqrt{(4k-(t-h)^2)^3}}{6k\pi} dt + h\mu_{n-1}$$

but

$$\left[\frac{-t^{n-1}\sqrt{(4k-(t-h)^2)^3}}{6k\pi} \right]_{h-2\sqrt{k}}^{h+2\sqrt{k}} = 0$$

so

$$\mu_n = -\frac{(n-1)}{3} \int_{h-2\sqrt{k}}^{h+2\sqrt{k}} \frac{t^{n-2}(h^2 - 4k + t^2 - 2ht)\sqrt{4k-(t-h)^2}}{2k\pi} dt + h\mu_{n-1} =$$

$$\begin{aligned}
&= -\frac{(h^2 - 4k)(n-1)}{3}\mu_{n-2} - \frac{(n-1)}{3}\mu_n + \frac{2h(n-1)}{3}\mu_{n-1} + h\mu_{n-1} = \\
&= -\frac{(h^2 - 4k)(n-1)}{3}\mu_{n-2} - \frac{(n-1)}{3}\mu_n + \frac{h(2n+1)}{3}\mu_{n-1}
\end{aligned}$$

we finally find the recurrence

$$\mu_n = \frac{h(2n+1)\mu_{n-1} - (h^2 - 4k)(n-1)\mu_{n-2}}{n+2}$$

while μ_0 and μ_1 can be easily found calculating (49) for $n = 0, 1$. \square

Corollary 5.2. *We have $\forall y \in \mathbb{R} \ L^{(y)}(\mu(h, k)) = \mu(h + y, k)$.*

Proof. In fact if

$$\mu'_n = \sum_{i=0}^n \binom{n}{i} y^{n-i} \mu_i$$

using (49) we have

$$\mu'_n = \int_{h+2\sqrt{k}}^{h+2\sqrt{k}} \sum_{i=0}^n \binom{n}{i} y^{n-i} t^i \frac{\sqrt{4k - (t-h)^2}}{2k\pi} dt$$

Now

$$\sum_{i=0}^n \binom{n}{i} y^{n-i} t^i = (t + y)^n$$

and substituting $u = t + y$ and $h' = h + y$

$$\mu'_n = \int_{h'-2\sqrt{k}}^{h'+2\sqrt{k}} \frac{u^n \sqrt{4k - (u-h')^2}}{2k\pi} du$$

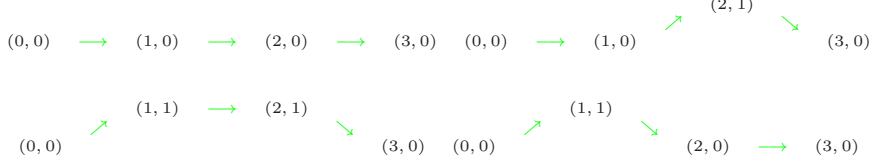
Thus μ'_n is defined with an analogous relation like (49) for μ_n . \square

6 Combinatorial interpretation for $\mu_n(h, k)$

We consider a lattice $(n+1) \times (n+1)$ composed by all the points having non negative integer coordinates. Motzkin paths are all the courses starting from $(0, 0)$ and reaching $(n, 0)$ with the following rules

$$\begin{cases} (i, j) \rightarrow (i+1, j) & \text{horizontal shift to east} \\ (i, j) \rightarrow (i+1, j+1) & \text{diagonal shift to north-east} \\ (i, j) \rightarrow (i+1, j-1) & \text{diagonal shift to south-east} \end{cases}$$

For example from $(0,0)$ to $(3,0)$ we have only the 4 possible paths



If we weight one shift of a path \mathcal{P} posing:

$$\begin{cases} w((i,j) \rightarrow (i+1,j)) = h \\ w((i,j) \rightarrow (i+1,j+1)) = 1 \\ w((i,j) \rightarrow (i+1,j-1)) = k \end{cases}$$

we can describe \mathcal{P} with weights product. The four paths represented above are respectively represented by: h^3 , hk , hk , kh . We observe that the sum of all the weights of these paths from $(0,0)$ to $(3,0)$ is $h^3 + 3hk = \mu_3(h,k)$. This is a consequence of the

Theorem 6.1 (Viennot's Theorem [12]). *Under the rules described above, for every Motzkin path P the following relation holds*

$$\mu_n(h,k) = \sum_{P:(0,0) \rightarrow (n,0)} w(P) \quad .$$

As a consequence in $\mu_n(h,k)$ is codified information about all weighted paths from $(0,0)$ to $(n,0)$:

- the sum of coefficients of $\mu_n(h,k)$ gives the number of all possible Motzkin paths from $(0,0)$ to $(n,0)$;
- the h exponent in every term gives the number of horizontal shifts to east;
- the k exponent in every term gives the number of diagonal shifts to north-east (or to south-east);
- the weight h may be interpreted as the number of colors among which we can select one to draw horizontal shifts to east;
- the weight k may be interpreted as the number of colors among which we can select one to draw diagonal shifts to north-east or to south-east;

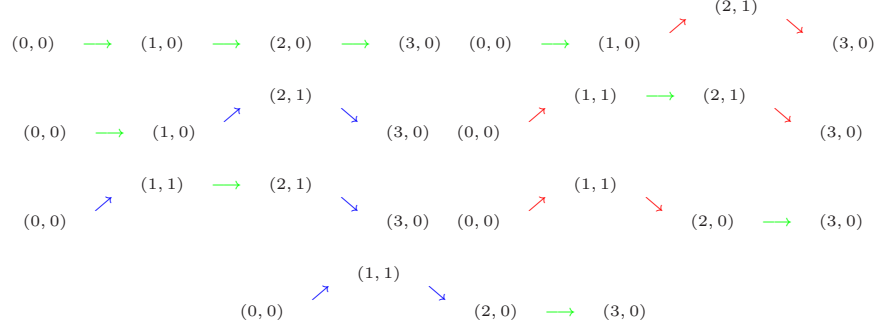
Example 6.2.

From $\mu_4(h,k) = h^4 + 6h^2k + 2k^2$ we have $9 = 1 + 6 + 2 = \mu_4(1,1)$ distinct paths from $(0,0)$ to $(4,0)$ traced with one color for all shifts:

- 1 path having 4 horizontal shifts;

- 6 with 2 horizontal shifts and 1 to north-east (and so 1 to south-east);
- 2 with 2 diagonal shifts to north-east (and so 2 to south-east).

Moreover from $\mu_3(1, 1) = 4$ we recover the previous one-colored paths and from $\mu_3(1, 2) = 7$ we find all the paths painted with one color for horizontal shifts and two possible colors for diagonal shifts:



7 The action of η

We begin with an example

Example 7.1. If we consider $\mu(h, k)$ (in this section we take , without loss of generality, $-h$ instead of h) we have from (38)

$$u = \frac{1 + ht - \sqrt{(1 + ht)^2 - 4kt^2}}{2kt}$$

which solved as an equation in t gives

$$t = \frac{u}{ku^2 - hu + 1} = \sum_{n=0}^{+\infty} F_n(h, k) u^{n+1}.$$

We used (30) with $b = h$ obtaining $F(h, k) = \{F_n(h, k)\}_{n=0}^{+\infty}$, the generalized Fibonacci sequence. So we note that

$$\begin{cases} \eta(F(h, k)) = \mu(h, k) \\ \eta(\mu(h, k)) = F(h, k) \end{cases}.$$

Observation 7.2. Recalling the Corollary 2.5 we have an alternative way to find the relation proved in Corollary 5.2. When $b = h$

$$\mathcal{W}(1, h, h, k) = F(h, k)$$

and from example (7.1)

$$I^{(x)}(\mu(h, k)) = \eta(\mathcal{W}(1, h - x, h - 2x, x^2 - hx + k))$$

$$L^{(x)}(\mu(h, k)) = \eta(\mathcal{W}(1, h - x, h - x, k)) = \eta(F(h - x, k)) = \mu(h - x, k)$$

The terms of $B = \eta(A)$, can be expressed by means of **Lagrange inversion formula** [7]

$$(51) \quad b_n = \frac{1}{(n+1)!} \frac{d^n}{du^n} \left\{ \left[\frac{u}{t(u)} \right]^{n+1} \right\} \Big|_{u=0}.$$

Using Lagrange inversion formula we can find an analogous expression of (43) for $\mu_n(h, k)$. In fact

$$t(u) = \frac{u}{ku^2 - hu + 1}$$

thus

$$\frac{u}{t(u)} = ku^2 - hu + 1$$

and

$$\mu_n(h, k) = b_n = \frac{1}{(n+1)!} \frac{d^n}{du^n} \{ (ku^2 - hu + 1)^{n+1} \} \Big|_{u=0}.$$

The trinomial expansion gives

$$(ku^2 - hu + 1)^{n+1} = \sum_{p+q+r=n+1} \frac{(n+1)!}{p!q!r!} (-hu)^r (ku^2)^q$$

and so

$$(ku^2 - hu + 1)^{n+1} = \sum_{p+q=0}^{n+1} \frac{(n+1)!}{p!(n+1-p-q)!q!} (-h)^{n+1-p-q} k^q u^{n+1-p+q}.$$

Differentiating n times

$$\frac{d^n}{du^n} \{ (ku^2 - hu + 1)^{n+1} \} = \sum_{p+q=0}^{n+1} \frac{(n+1)!(n+1-p+q)!}{p!(n+1-p-q)!(q-p+1)!q!} (-h)^{n+1-p-q} k^q u^{q-p+1}.$$

For $u = 0$ the only non zero term occurs when $q = p - 1$. Consequently $p + q = 2p - 1$ and $0 \leq p + q \leq n + 1$ implies $1 \leq p \leq \lfloor \frac{n+2}{2} \rfloor$ then

$$\mu_n(h, k) = \sum_{p=1}^{\lfloor \frac{n+2}{2} \rfloor} \frac{n!}{p!(n-2p+2)!(p-1)!} (-h)^{n-2p+2} k^{p-1}.$$

Taking $h = -x$ and considering odd and even values for n , we have

$$(52) \quad \begin{cases} \mu_n(-x, k) = \sum_{j=0}^{\frac{n-1}{2}} A_j^{(n)} x^{2j+1} & n \text{ odd} \\ \mu_n(-x, k) = \sum_{j=0}^{\frac{n}{2}} A_j^{(n)} x^{2j} & n \text{ even} . \end{cases}$$

where

$$(53) \quad A_j^{(n)} = \begin{cases} \frac{1}{n+1} \begin{pmatrix} n+1 \\ \frac{n+1}{2} - j, \quad 2j+1, \quad \frac{n-1}{2} - j \end{pmatrix} k^{\frac{n-1}{2}-j} & n \text{ odd} \\ \frac{1}{n+1} \begin{pmatrix} n+1 \\ \frac{n}{2} + 1 - j, \quad 2j, \quad \frac{n}{2} - j \end{pmatrix} k^{\frac{n}{2}-j} & n \text{ even} . \end{cases}$$

Observation 7.3.

If $x = 0$ for $n = 2m$ we have

$$\mu_{2m}(0, k) = k^m C_m \quad C_m = \frac{1}{2m+1} \binom{2m+1}{m}$$

and C_m is the m -th Catalan number, while if $n = 2m+1$ we have

$$\mu_{2m+1}(0, k) = 0 \quad .$$

Observation 7.4 (Orthogonality relations). We consider the polynomial $P_n(x) = \mathcal{W}(1, h+x, h+x, k)$. Its explicit expression can be found observing that from (30) we have

$$\sum_{n=0}^{+\infty} P_n(x) t^n = \frac{1}{1 - (h+x)t + kt^2} = \sum_{j=0}^{+\infty} ((h+x)t - kt^2)^j = \sum_{j=0}^{+\infty} \sum_{l=0}^j \binom{j}{l} (h+x)^{j-l} (-k)^l t^{l+j} \quad .$$

Rearranging indexes and posing $l+j=n$ we obtain

$$(54) \quad P_n(x) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-l}{l} (h+x)^{n-2l} (-k)^l .$$

and the generic coefficient $P_j^{(n)}$ of x^j follows from the j -th derivative:

$$(55) \quad P_j^{(n)} = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-l}{l} \binom{n-2l}{j} h^{n-2l-j} (-k)^l \quad .$$

Now from definition of the functional \mathcal{V} [5]

$$(56) \quad \begin{cases} \mathcal{V}[1] = 1 \\ \mathcal{V}[P_m(x)P_n(x)] = 0 \text{ if } m \neq n \end{cases}$$

from (37) $P_0(x) = 1$ and from (56) choosing $m = 0$ we have

$$\mathcal{V}[P_n(x)] = \delta(n, 0)$$

which in this case becomes the following relation

$$\sum_{j=0}^n P_j^{(n)} \mu_j(h, k) = \sum_{j=0}^n \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-l}{l} \binom{n-2l}{j} h^{n-2l-j} (-k)^l \sum_{p=1}^{\lfloor \frac{j+2}{2} \rfloor} \frac{j! (-h)^{j-2p+2} k^{p-1}}{p!(j-2p+2)!(p-1)!} = \delta(n, 0) \quad .$$

And when $h = 0$ we have

$$(57) \quad P_n(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (-k)^i x^{n-2i} = E_n(x, k) \quad .$$

where $E_n(x, k)$ is the n -th Dickson polynomial of the second kind [8] .

So if $n = 2m$, recalling that $\mu_{2m}(0, k) = k^m C_m$, we obtain a similar orthogonality relation where Catalan numbers are involved

$$(58) \quad \sum_{i=0}^m \binom{2m-i}{i} (-k)^i k^{m-i} C_{m-i} = \delta(m, 0) \quad .$$

As far as we know this Catalan identity is new. Of course (58) is not difficult to prove (try it with Zeilberger's program [9], for example), but it seems interesting also for the context it rises from.

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